Lecture Notes, Lectures 14, 15,
6.1. Firms, Profits, and Household Income

$\alpha^{ij} \in R_+ =$ i's share of firm j

$\sum_{i \in H} \alpha^{ij} = 1$ for each $j \in F$, and $\alpha^{ij} \geq 0$, for all $i \in H$, $j \in F$.

$$r \equiv \sum_{i \in H} r^i \ .$$

For each $j \in F$, $\tilde{\pi}^j(p) = \max_{y \in \mathcal{Y}^j} p \cdot y = p \cdot \tilde{S}^j(p)$

Theorem 6.1: Assume P.II, P.III,P.V, P.VI. $\tilde{\pi}^j(p)$ is a well-defined continuous function of p for all $p \in R^N_+$, $p \neq 0$.

Define $\tilde{M}^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \tilde{\pi}^j(p)$ (homogeneous of degree one in p)

Under P.II, P.III, P.V,P.VI, $\tilde{M}^i(p)$ is continuous, real valued, nonnegative and well defined for all $p \in R^N_+$, $p \neq 0$. By Lemma 5.2 ---- using this definition of $\tilde{M}^i(p)$ ---- $\tilde{B}^i(p)$ and $\tilde{D}^i(p)$ are homogeneous of degree 0 in p. Then we can restrict the price space to

$$P = \left\{ p \middle| p \in R^N,\ p_k \geq 0,\ k = 1\ldots, N,\ \sum_{k=1}^{N} p_k = 1 \right\}.$$

6. 2. Excess Demand and Walras' Law

$$\tilde{Z}(p) = \tilde{D}(p) - \tilde{S}(p) - r = \sum_{i \in H} \tilde{D}^i(p) - \sum_{j \in F} \tilde{S}^j(p) - \sum_{i \in H} r^i .$$

Lemma 6.1: Assume C.I - C.V, C.VII,C.VIII, and P.I , P.III, P.V, P.VI. The range of $\tilde{Z}(p)$ is bounded. $\tilde{Z}(p)$ is continuous and well defined for all p $\in$ P.

Proof: Apply Theorems 5.2 and 4.1. The finite sum of bounded sets is bounded. The finite sum of continuous functions is continuous.

Theorem 6.2 (Weak Walras' Law) : Assume C.I - C.V, C.VII,C.VIII, and P.I, P.III, P.V, P.VI. For all p $\in$ P, $p \cdot \tilde{Z}(p) \leq 0$. For p such that $p \cdot \tilde{Z}(p) < 0$, there is k = 1, 2, ... , N so that $\tilde{Z}_k(p) > 0$.

Intuition for the (weak) Walras Law

$p \cdot \tilde{Z}(p) = p \cdot \tilde{D}(p) - p \cdot \tilde{S}(p) - p \cdot r$. The expression $p \cdot \tilde{D}(p)$ is the value of total household spending in the economy. The expressions $p \cdot \tilde{S}(p)$, $p \cdot r$ are total profits and total value of household endowment; their sum is the value of total household income. So the (weak) Walras Law merely says that total household expenditure is limited by total household income and is usually equal to income (unless length of the demand vector is a binding constraint).

Proof: For each i in H, $p \bullet \tilde{D}^i(p) \leq \tilde{M}^i(p) = p \bullet r^i + \sum_{j \in F} \alpha^{ij} \tilde{\pi}^j(p)$.

$\sum_{i \in H} \alpha^{ij} = 1$ for each $j \in F$.

$$p \bullet \tilde{Z}(p) = p \bullet [\sum_{i \in H} \tilde{D}^i(p) - \sum_{j \in F} \tilde{S}^j(p) - \sum_{i \in H} r^i]$$

$$= \ p \bullet \sum_{i \in H} \tilde{D}^i(p) - p \bullet \sum_{j \in F} \tilde{S}^j(p) - p \bullet \sum_{i \in H} r^i$$

$$= \sum_{i \in H} p \cdot \tilde{D}^i(p) - \sum_{j \in F} p \cdot \tilde{S}^j(p) \ - \sum_{i \in H} p \cdot r^i$$

$$= \sum_{i \in H} p \cdot \tilde{D}^i(p) - \sum_{j \in F} \tilde{\pi}^j(p) \ - \sum_{i \in H} p \cdot r^i$$

$$= \sum_{i \in H} p \cdot \tilde{D}^i(p) - \sum_{j \in F} [\sum_{i \in H} \alpha^{ij} \tilde{\pi}^j(p)] \ - \sum_{i \in H} p \cdot r^i$$

$$= \sum_{i \in H} p \cdot \tilde{D}^i(p) - \sum_{i \in H} [\sum_{j \in F} \alpha^{ij} \tilde{\pi}^j(p)] \ - \sum_{i \in H} p \cdot r^i$$

$= \sum_{i \in H} p \cdot \tilde{D}^i(p) - \sum_{i \in H} \tilde{M}^i(p) \leq 0$, in asmuch as $p \bullet \tilde{D}^i(p) \leq \tilde{M}^i(p)$ for each i.

Suppose p • $\tilde{Z}$ (p) < 0. Then
$$p \cdot \sum_{i \in H} \tilde{D}^i(p) < p \cdot r + p \cdot \sum_{j \in F} \tilde{S}^j(p) = \sum_{i \in H} \tilde{M}^i(p) ,$$
so for some i ∈ H, p•$\tilde{D}^i$(p) < $\tilde{M}^i$(p) .

Now we apply Lemma 5.3. By weak monotonicity, C. IV, p•$\tilde{D}^i$(p) = $\tilde{M}^i$(p) or $|\tilde{D}^i(p)|$ = c. Hence we must have $|\tilde{D}^i(p)|$ = c. Recall that c is chosen so that |x| < c (a strict inequality) for all attainable x. But then $\tilde{D}^i$(p) is not attainable. For <u>no</u> y ∈ $\mathcal{Y}$ do we have $\tilde{D}^i(p) \leq$ y + r . Therefore $\tilde{Z}_k$(p) > 0, some k = 1, 2, ..., N.

QED

2.7 Brouwer Fixed Point Theorem

Definition: Let $x_1, x_2, \ldots, x_{N+1}$ be N+1 (any N of them should be linearly independent) points in R^N. Then the <u>N-simplex</u> defined by $x_1, \ldots, x_{N+1}$ is the set S of convex combinations of $x_1, x_2, \ldots, x_{N+1}$

$$S \equiv \left\{ x \mid x = \sum_{i=1}^{N+1} \lambda_i x_i,\ \lambda_i \geq 0,\ \sum_{i=1}^{N+1} \lambda_i = 1 \right\}.$$

Theorem 2.10 (Brouwer Fixed Point Theorem): Let S be an N-simplex in R^N and let $f: S \to S$, f continuous. Then there is $x^* \in S$, so that $f(x^*) = x^*$.

Note that compactness and convexity (at least no holes in the middle) of S and continuity of f are all essential to the BFPT.